

Discrete $(n + 1)$ -valued states and n -perfect pseudo-effect algebras

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Abstract

We give sufficient and necessary conditions to guarantee that a pseudo-effect algebra admits an $(n + 1)$ -valued discrete state. We introduce n -perfect pseudo-effect algebras as algebras which can be split into $n + 1$ comparable slices. We prove that the category of strong n -perfect pseudo-effect algebras is categorically equivalent to the category of torsion-free directed partially ordered groups of a special type.

Keywords: Pseudo-effect algebra; po-group; symmetric pseudo-effect algebra; $(n + 1)$ -valued discrete state; n -perfect pseudo-effect algebra; lexicographic product; categorical equivalence

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1 Introduction and basic definitions

Effect algebras were introduced by Foulis and Bennett in 1994 for modeling unsharp measurements of a quantum mechanical system [20]. They are a generalization of many structures which arise in quantum mechanics, in particular of orthomodular lattices in noncommutative measure theory and of MV-algebras in fuzzy measure theory [16, 25, 27]. Alternative structures called difference posets, which are categorically equivalent with effect algebras, were introduced in [26]. At the end of the 90's, a noncommutative version of MV-algebras, called pseudo-MV-algebras, was introduced in [22] and independently in [28] as generalized MV-algebras, GMV-algebras. A noncommutative generalization of effect algebras, called pseudo-effect algebras, PEAs, was introduced and studied in [17, 18].

Perfect MV-algebras, introduced by Belluce, Di Nola and Lettieri in [2], may be viewed as the most compelling examples of non-Archimedean MV-algebras, in such a sense that they are generated by their infinitesimals. In a perfect MV-algebra, every element belongs either to its radical or its coradical. In [7], it is shown that for any perfect MV-algebra M , there exists an Abelian unital ℓ -group G (lattice-ordered group), such that M is isomorphic to an interval of the lexicographic product of the group of integers \mathbb{Z} and G .

Especially, MV-algebras are lattice-ordered effect algebras satisfying the Riesz Decomposition Property (RDP). For any effect algebra E with (RDP), there exists a partially ordered group G

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with a strong unit u such that E is isomorphic to the effect algebra $\Gamma(G, u) := [0, u]$, where the effect algebra operations are the group additions existing in $[0, u]$. In [13], Dvurečenskij introduced perfect effect algebras, which are one kind of effect algebras admitting (RDP), and he proved that every perfect algebra is an interval in the lexicographical product of the group of integers with an Abelian directed partially ordered group with interpolation. Moreover, Dvurečenskij showed that the category of perfect effect algebras is categorically equivalent to the category of Abelian directed partially ordered groups with interpolation.

The principal result on a representation of GMV-algebras says that every GMV-algebra is always an interval in a unital ℓ -group [9], i.e. an ℓ -group with strong unit. This result provides a new bridge between different research areas, including GMV-algebras, unital ℓ -groups, noncommutative many valued logic, soft computing and quantum structures [16]. Especially, using this result, perfect GMV-algebras and n -perfect GMV-algebras were introduced in [6, 14]. Furthermore, the author proved that any n -strong perfect GMV-algebra is always an interval in the lexicographical product of the group of integers \mathbb{Z} with an ℓ -group.

The notion of a state, as an analogue of a probability measure, is crucial for quantum mechanical measurements. Therefore, a special attention is done in order to exhibit whether does a state exist for the studied structure and if yes, what are its basic properties. In any Boolean algebra, we have a lot of two-valued states, and in general, every two valued state is extremal. Every perfect MV-algebra or every perfect effect algebra admits only a two-valued state. On the other hand, in the orthodox example of quantum mechanics, see e.g. [8], the system $L(H)$ of all closed subspaces of a Hilbert space H , $\dim H \geq 3$, does not admit any two-valued state. A two-valued state is a special case of an $(n + 1)$ -valued discrete state, and in this paper we concentrate to the existence of such states. We recall that, if $\dim H = n$, then $L(H)$ admits a unique $(n + 1)$ -valued discrete state s , namely $s(M) = \dim M/n$, $M \in L(H)$. In general, discrete states are of particular importance in many areas of mathematics, see [24]. For example, discrete states were used in [4, Thm 3.2] in order to completely describe monotone σ -complete effect algebras.

GMV-algebras are also PEAs, and so, n -perfect GMV-algebras are PEAs. Any n -perfect GMV-algebra admits an $(n + 1)$ -valued discrete state. The main accent of the present paper is done to the study of $(n + 1)$ -valued discrete states on PEAs and to show how they are related with n -perfect PEAs.

Therefore, we start with the study of the structure of PEAs admitting a two-valued state. Then we present one characterization of PEAs admitting an $(n + 1)$ -valued discrete state. Later, we give a definition of n -strong perfect PEAs, and we prove that any n -strong perfect GMV-algebra is always an interval in the lexicographical product of the group of integers \mathbb{Z} with a torsion-free partially ordered group G such that $\mathbb{Z} \overrightarrow{\times} G$ satisfies a special type of the Riesz Decomposition Property. Finally, the relationships between the category of n -strong perfect PEAs and the category of torsion-free directed partially ordered groups of a special type are discussed.

The paper is organized as follows. Some basic definitions and properties about pseudo-effect algebras are presented in Section 2. In Section 3, we study the structure of pseudo-effect algebras with two-valued states. In Section 4, we give sufficient and necessary conditions to guarantee that

a pseudo-effect algebra admits an $(n + 1)$ -valued discrete state. In Section 5, we introduce the class of n -perfect pseudo-effect algebras. In Section 6, we study the class of strong n -perfect pseudo-effect algebras and we prove that the category of strong n -perfect pseudo-effect algebras is categorically equivalent to the category of torsion-free directed partially ordered groups of a special type.

2 Basic definitions and facts

In this section, we give basic definitions and facts about pseudo-effect algebras which we will need in this paper.

Definition 2.1. [19] A structure $(E; +, 0)$, where $+$ is a partial binary operation and 0 is a constant, is called a *generalized pseudo-effect algebra*, or GPEA for short, if for all $a, b, c \in E$, the following hold.

- (GP1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
- (GP2) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- (GP3) If $a + b$ and $a + c$ exist and are equal, then $b = c$. If $b + a$ and $c + a$ exist and are equal, then $b = c$.
- (GP4) If $a + b$ exists and $a + b = 0$, then $a = b = 0$.
- (GP5) $a + 0$ and $0 + a$ exist and both are equal to a .

According to [19], we introduce a binary relation \leq in a GPEA E . For $a, b \in E$, we define $a \leq b$ if and only if there is an element $c \in E$ such that $a + c = b$. Equivalently, there exists an element $d \in E$ such that $d + a = b$. Then \leq is a partial order on E .

We introduce two partial binary operations $/$ and \backslash on a GPEA E . For any $a, b \in E$, a/b is defined if and only if $b \backslash a$ is defined if and only if $a \leq b$, and in such a case we have $(b \backslash a) + a = a + (a/b) = b$. Then $a = (b \backslash a)/b = b \backslash (a/b)$.

By [32], a GPEA E is called a *weakly commutative* GPEA if it satisfies the following condition:

- (C) for any $a, b \in E$, $a + b$ exists if and only if $b + a$ exists .

Definition 2.2. [17] A structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra*, or PEA for short, if for all $a, b, c \in E$, the following hold.

- (PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
- (PE2) There are exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.
- (PE3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- (PE4) If $a + 1$ or $1 + a$ exists, then $a = 0$.

Let a be an element of a PEA E and $n \geq 0$ be an integer. We define $na = 0$ if $n = 0$, $1a = a$ if $n = 1$, and $na = (n-1)a + a$ if $(n-1)a$ and $(n-1)a + a$ are defined in E . We define the *isotropic index* $\iota(a)$ of the element a , as the maximal nonnegative number n such that na exists. If na exists for every integer n , we say that $\iota(a) = +\infty$. In the following, we denote by $\text{Infinit}(E) = \{a \in E \mid \iota(a) = +\infty\}$.

We recall that if $(E; +, 0, 1)$ is a PEA, then $(E; +, 0)$ is a GPEA. If $a + b$ exists and $a + b = 1$, then we write $b^- = a$, $a^\sim = b$. Thus, two mappings $a \mapsto a^-$ and $a \mapsto a^\sim$ are unary operations satisfying the following:

- (i) if $a \leq b$, then $b^- \leq a^-$, $b^\sim \leq a^\sim$.
- (ii) for any $a \in E$, $a^{-\sim} = a^{\sim-} = a$.

Assume that $(G; +, \leq, 0)$ is a po-group (po-group for short), i.e. G is a group written additively with a partial ordering \leq such that $a \leq b$ implies $c + a + d \leq c + b + d$ for any $c, d \in G$. A positive element $u \in G$ is said to be a *strong unit* if, given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$. The pair (G, u) is said to be a *unital po-group*. A po-group is said to be *directed* if, $a, b \in G$, there is an element $c \in G$ such that $a, b \leq c$. For more about po-groups, see [23].

We denote by $G^+ := \{g \in G \mid g \geq 0\}$. For any $x, y \in G^+$, let $x + y$ be the group addition of x and y . Then $(G^+; +, 0)$ is a generalized pseudo-effect algebra. We set $\Gamma(G, u) := \{x \in G \mid 0 \leq x \leq u\}$, and we endow $\Gamma(G, u)$ with the operation $+$ such that $a + b$ is defined in $\Gamma(G, u)$ whenever $a \leq u - b$, and in such a case, $a + b$ in $\Gamma(G, u)$ is the group addition of a and b . Then $(\Gamma(G, u), +, 0, u)$ is a pseudo-effect algebra. We say that a PEA E is an *interval PEA* if there exists a po-group G such that E is isomorphic to a PEA $\Gamma(G, u)$ for some strong unit $u \in G^+$.

We say that a PEA E satisfies the *Riesz Decomposition Property* (RDP), if $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{12} + c_{22}$. If, in addition, for all $x \leq c_{12}$ and $y \leq c_{21}$, we have $x + y, y + x$ exists in E and $x + y = y + x$, we say that E satisfies $(\text{RDP})_1$. By [18], every PEA E with $(\text{RDP})_1$ is an interval PEA.

A PEA E satisfies $(\text{RDP})_0$ if, for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$, there are $d_1, d_2 \in E$ such that $d_1 \leq b_1$, $d_2 \leq b_2$ and $a = d_1 + d_2$.

We recall that $(\text{RDP})_1 \Rightarrow (\text{RDP}) \Rightarrow (\text{RDP})_0$, but the converse is not true, see [18].

Finally, we say a directed po-group G satisfies (RDP) or $(\text{RDP})_1$ if the same property as for PEA holds also for the positive cone G^+ .

We recall that according to [18, Thm 5.7], for any PEA E with $(\text{RDP})_1$ there is a unique (up to isomorphism of unital po-groups) unital po-group (G, u) , where G satisfies $(\text{RDP})_1$, such that $E \cong \Gamma(G, u)$. Moreover, there is a categorical equivalence between the category of PEAs with $(\text{RDP})_1$ and the category of unital po-groups (G, u) , where G^+ satisfies $(\text{RDP})_1$.

Example 2.3. Let \mathbb{Z} be the group of integers and $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Define for every two elements of G

$$(a, b, c) + (x, y, z) = \begin{cases} (a + x, b + y, c + z), & x \text{ is even,} \\ (a + x, c + y, b + z), & x \text{ is odd,} \end{cases}$$

and define $(a, b, c) \leq (x, y, z)$ if $a < x$ or $a = x, b \leq y$ and $c \leq z$.

Then $(G; +, \leq)$ is a lattice-ordered group with strong unit $u = (1, 0, 0)$, and $\Gamma(G, (1, 0, 0))$ is a PEA.

Proposition 2.4. *Let E be a PEA. Then E is a weakly commutative if and only if, for any $a \in E$, $a^- = a^\sim$.*

Proof. Assume that, for any $a \in E$, we have that $a^- = a^\sim$. If $a + b$ exists in E for $a, b \in E$, then $b \leq a^\sim = a^-$, which implies that $b + a$ exists in E . Hence, E is a weakly commutative PEA.

Conversely, assume that E is a weakly commutative PEA. Then for any $a \in E$, $a^- + a$ exists and equals the unit 1. By the condition (C), we have that $a + a^-$ exists in E , and so $a^- \leq a^\sim$. Since the equality $a + a^\sim = 1$ holds, we have that $a^\sim + a$ exists, and so $a^\sim \leq a^-$. Hence, we conclude $a^\sim = a^-$. \square

Remark 2.5. In [11], Dvurečenskij has introduced symmetric pseudo-effect algebras as following: a pseudo-algebra E is said to be *symmetric* (or, more precisely, with symmetric differences) if $a^- = a^\sim$ for any $a \in E$. Now, by Proposition 2.4, the set of weakly commutative pseudo-effect algebras coincides with the set of symmetric pseudo-effect algebras. In such a case, we set $a' = a^- = a^\sim$ and a' is said to be an orthosupplement of a .

Example 2.6. Let \mathbb{Z} be the group of integers and G be a (not necessarily Abelian) po-group. Let $\mathbb{Z} \vec{\times} G$ be the lexicographic product of \mathbb{Z} and G , then $\mathbb{Z} \vec{\times} G$ is a directed po-group with strong unit $(1, 0)$. If we set $E = \Gamma(\mathbb{Z} \vec{\times} G, (1, 0))$, then E is a symmetric PEA but not necessarily commutative PEA.

Let $(G; +, \leq, 0)$ be a po-group. An element $c \in G$ such that $x + c = c + x$ for all $x \in G$ is said to be a commutator of G . We set $C(G) = \{c \in G \mid c \text{ is a commutator of } G\}$.

Example 2.7. Let G be a po-group. Assume that $c \in G^+$ is a commutator, then $E = \Gamma(\mathbb{Z} \vec{\times} G, (n, c))$ is a symmetric PEA. Especially, $\Gamma(\mathbb{Z} \vec{\times} G, (n, 0))$ is a symmetric PEA.

Let $(E; +, 0)$ be a GPEA. Let E^\sharp be a set disjoint from E with the same cardinality. Consider a bijection $a \mapsto a^\sharp$ from E onto E^\sharp and let us denote $E \cup E^\sharp = \hat{E}$. Define a partial operation $+^*$ on \hat{E} by the following rules. For $a, b \in E$,

- (i) $a +^* b$ is defined if and only if $a + b$ is defined, and $a +^* b := a + b$.
- (ii) $a +^* b^\sharp$ is defined if and only if $b \setminus a$ is defined, and then $a +^* b^\sharp := (b \setminus a)^\sharp$.
- (iii) $b^\sharp +^* a$ is defined if and only if a/b is defined, and then $b^\sharp +^* a := (a/b)^\sharp$.

In [32], we have obtained the following results.

Proposition 2.8. *If $(E; +, 0)$ is a symmetric GPEA, then the structure $(\hat{E}; +^*, 0, 0^\sharp)$ is a symmetric PEA. Moreover, E is an order ideal in \hat{E} closed under $+$, and the partial order induced by $+^*$, when restricted to E , coincides with the partial order induced by $+$.*

Proposition 2.9. *Let $(E; +, 0)$ be a GPEA and let the structure $(\hat{E}; +^*, 0, 0^\sharp)$ be a PEA, then $(E; +, 0)$ is a symmetric GPEA and $(\hat{E}; +^*, 0, 0^\sharp)$ is a symmetric PEA.*

By Proposition 2.8 and 2.9, we immediately conclude the following result.

Corollary 2.10. *Let $(E; +, 0)$ be a GPEA. Then the algebraic system $(\hat{E}; +^*, 0, 0^\sharp)$ is a PEA if and only if $(E; +, 0)$ is a symmetric GPEA.*

The symmetric PEA $(\hat{E}; +^*, 0, 0^\sharp)$ is usually called the *unitization* of a symmetric GPEA $(E; +, 0)$, and for any $a \in E$, $a + a^\sharp = a^\sharp + a = 0^\sharp$, hence, $a' = a^\sharp$ and $a^{\sharp'} = a$. Since the operation $+^*$ on \hat{E} coincides with the $+$ operation on E , it will cause no confusion if we use the notation $+$ also for its extension on \hat{E} .

Definition 2.11. (i) Let E, F be two GPE-algebras. A mapping $f : E \rightarrow F$ is a *morphism* if the following conditions are satisfied:

- (1) $f(0_E) = 0_F$.
- (2) If $a, b \in E$ and $a + b$ exists, then $f(a) + f(b)$ exists and $f(a + b) = f(a) + f(b)$.

(ii) Let E, F be two pseudo-effect algebras. A mapping $f : E \rightarrow F$ is a *morphism* if the following conditions are satisfied:

- (1) $f(0_E) = 0_F, f(1_E) = 1_F$.
- (2) If $a, b \in E$ and $a + b$ exists, then $f(a) + f(b)$ exists and $f(a + b) = f(a) + f(b)$.

(iii) Let E be a PEA. Then any morphism $s : E \rightarrow [0, 1]$ is said to be a *state* on E . A state s is said to be *discrete* if there exists an integer n such that $s(E) \subseteq \{0, \frac{1}{n}, \dots, 1\}$, where $s(E) = \{s(x) \mid x \in E\}$. If $s(E) = \{0, \frac{1}{n}, \dots, 1\}$, we say that s is an $(n + 1)$ -valued discrete state.

Especially, if $n = 1$, then we say that s is a *two-valued state*.

(iv) A state s is said to be *extremal* if, for any states s_1, s_2 and $\alpha \in (0, 1)$, the equation $s = \alpha s_1 + (1 - \alpha)s_2$ implies $s = s_1 = s_2$.

Of course, every two-valued state is a 2-valued discrete state, and vice-versa. For example, every two-valued state on a PEA E is extremal.

We note that if s is a state on E , then $s(0) = 0$ and $s(1) = 1$, therefore, in what follows we will assume $0 \neq 1$.

Example 2.12. *Let G be a directed po-group and let $c \in G$. Then $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, c))$ admits an $(n + 1)$ -valued discrete state.*

We recall that the real interval $[0, 1]$ can be assumed also as an interval effect algebra $\Gamma(\mathbb{R}, 1)$.

Remark 2.13. Let E be a PEA and $s : E \rightarrow [0, 1]$ be a state with $|s(E)| = n + 1$.

(i) If $n = 1$, then $s(E) = \{0, 1\}$ is a sub-effect algebra of $[0, 1]$ and s is a two-valued state.

(ii) If $n > 1$, then $s(E)$ is not necessarily a sub-effect algebra of $[0, 1]$. For example, let $E = \{0, a, b, 1\}$, we endow E with the partial operation $+$ as follows, (1) for any $x \in E$, $x + 0 = 0 + x = x$, (2) $a + b = b + a = 1$. Then the algebraic system $(E; +, 0, 1)$ is an effect algebra. We define a mapping $s : E \rightarrow [0, 1]$ as follows, $s(0) = 0$, $s(a) = \frac{2}{5}$, $s(b) = \frac{3}{5}$, $s(1) = 1$, then s is a discrete state on E and $s(E) = \{0, \frac{2}{5}, \frac{3}{5}, 1\}$. However, $s(E)$ is not a sub-effect algebra of $[0, 1]$.

(iii) Let s be an $(n + 1)$ -valued discrete state on an effect algebra. Then s is not necessarily extremal. For example, for the effect algebra E in (ii), set $s(0) = 0$, $s(a) = s(b) = \frac{1}{2}$, $s(1) = 1$, then s is a 3-valued discrete state which is not extremal. In fact, we set $s_1(0) = s_1(a) = 0$, $s_1(b) = s_1(1) = 1$,

and $s_2(0) = s_2(b) = 0$, $s_2(a) = s_2(1) = 1$, then s_1, s_2 are two states on E , and $s = \frac{1}{2}s_1 + \frac{1}{2}s_2$, however, $s \neq s_1$, $s \neq s_2$. In [12, Prop 8.5], Dvurečenskij has proved that if s is an extremal discrete state on an effect algebra E with (RDP), then s is an $(n+1)$ -valued discrete state.

Theorem 2.14. *Let E be a PEA and $s : E \rightarrow [0, 1]$ be a state. Assume that $|s(E)| = n+1$ and $n \geq 1$. Then the following statements are equivalent.*

- (i) s is an $(n+1)$ -valued discrete state.
- (ii) $s(E)$ is a sub-effect algebra of the effect algebra $[0, 1]$.
- (iii) For any $t, u \in s(E)$, if $t \leq u$, then there exists a $v \in s(E)$, such that $t + v = u$.

Proof. If $n = 1$, then using Remark 2.13, it is easy to see that (i), (ii) and (iii) are mutually equivalent.

Now, we assume that $n > 1$ and $s(E) = \{0, t_1, \dots, t_{n-1}, 1\}$, where $0 < t_1 < t_2 < \dots < t_{n-1} < 1$.

(i) \Rightarrow (ii). If s is an $(n+1)$ -valued discrete state, then $s(E) = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ is a sub-effect algebra of $[0, 1]$.

(ii) \Rightarrow (iii). Assume that $s(E)$ is a sub-effect algebra of the effect algebra $[0, 1]$. For any $t, u \in s(E)$, if $t \leq u$, then there exists a $v \in s(E)$ such that $t + v$ exists and $t + v = u$.

(iii) \Rightarrow (ii). By (iii), for any $t, v \in s(E)$ with $t \leq v$, we have that $v - t \in s(E)$. We define a partial binary operation $+$ on $s(E)$ as follows: $t + v$ exists in $s(E)$ iff $t \leq 1 - v$, and then $t + v$ is the classical addition of two real numbers t and v . It is routine to verify that $(s(E); +, 0, 1)$ is an effect algebra. Further, for any $t, v \in s(E)$, $t + v$ exists iff $t + v \leq 1$, which implies that $(s(E); +, 0, 1)$ is a sub-effect algebra of $[0, 1]$.

(ii) \Rightarrow (i). Assume that (ii) holds, and so (iii) holds, too. It suffices to prove that $t_i = \frac{i}{n}$ for any $i \in \{1, \dots, n-1\}$.

If $n = 2$, then we have that $s(E) = \{0, t_1, 1\}$. By $0 < t_1 < 1$, there exists a real number $t \in s(E)$ such that $t + t_1 = 1$. Obviously, $t \neq 0, 1$, and so $t = t_1$, which implies that $t_1 = \frac{1}{2}$. Then (i) holds.

Now, we assume that $n > 2$. We are claiming that for any $i \in \{1, \dots, n-1\}$, $t_i = it_1$.

If $i = 2$, then $t_1 < t_2$, and there exists a $j \in \{1, 2\}$ such that $t_1 + t_j = t_2$, and so $j = 1$, hence $t_2 = 2t_1$.

Assume by induction that, for any $j \leq i < n-1$, we have proved $t_j = jt_1$. Since $t_i < t_{i+1}$, we have $t_{i+1} - t_i \in \{t_1, \dots, t_i\}$. If it would be $t_{i+1} - t_i \geq 2t_1$, then $t_i < t_i + t_1 < t_i + 2t_1 \leq t_i + t_j = t_{i+1}$ which is impossible because between t_i and t_{i+1} there is no element in $s(E)$. Hence, $t_{i+1} - t_i = t_1$ which proves $t_i = it_1$ for any $i = 1, \dots, n-1$.

Finally, $0 < 1 - t_{n-1} < \dots < 1 - t_1 < 1$ which gives $1 - t_{n-1} = t_1$ so that $t_1 = \frac{1}{n}$, and s is an $(n+1)$ -valued discrete state.

For interest, we also give another proof. We are assuming $s(E)$ is a sub-effect algebra of $[0, 1]$. Noticing that $\Gamma(\mathbb{R}, 1) = [0, 1]$, and so, by [3, Thm 2.4], there exists a subgroup G of \mathbb{R} such that $\Gamma(G, 1) = s(E)$. By [24, Lem 4.21], there are following two cases:

(a) G is a dense subgroup of \mathbb{R} . Then $|\Gamma(G, 1)| = |s(E)|$ is infinite, which is a contradiction with our assumption.

(b) G is a cyclic subalgebra of \mathbb{R} . Assume that G is generated by a positive element t , and so $G = \{nt \mid n \in \mathbb{Z}\}$. Thus, by $\Gamma(\mathbb{R}, 1) \subseteq [0, 1]$, we have that $t \in (0, 1)$, and $nt = 1$. In fact, by $1 \in G$, there exists a natural number m such that $mt = 1$. Thus, we have that $\Gamma(G, 1) = \{0, \frac{1}{m}, \dots, 1\}$, which implies that $s(E) = \{0, \frac{1}{m}, \dots, 1\}$. However, $|s(E)| = n+1$, and so $m = n$, hence, $s(E) = \{0, \frac{1}{n}, \dots, 1\}$.

Thus, we have proved that $s(E) = \{0, \frac{1}{n}, \dots, 1\}$. \square

3 Pseudo-effect algebras with two-valued states

In this section, we will study the structure of pseudo-effect algebras with two-valued states. We will prove that a pseudo-effect algebra E admits a two-valued state if and only if there exists an ideal I such that $E = I \cup I^- = I \cup I^\sim$, where $I^- = \{i^- \mid i \in I\}$, $I^\sim = \{i^\sim \mid i \in I\}$ and $I \cap I^- = I \cap I^\sim = \emptyset$.

We recall that a nonempty subset I of a GPEA E is called an *ideal* if the following conditions hold:

- (i) for any $a \in E$ and $i \in I$ with $a \leq i$, we have $a \in I$;
- (ii) for any $i, j \in I$ if $i + j$ exists in E , then we have $i + j \in I$.

If a set I is an ideal of a PEA E and $1 \notin I$, then the ideal I is called *proper*.

An ideal I in a GPEA E is called *normal* if, for any $a, i, j \in E$ such that $a + i$ and $j + a$ exist and are equal, we have $i \in I$ if and only if $j \in I$.

For example, if s is a state of a PEA E , then the set $\text{Ker}(s) = \{x \in E \mid s(x) = 0\}$, *kernel* of s , is a normal ideal of E .

An ideal I in a GPEA E is called *maximal* if it is a proper ideal of E and is not included properly in any proper ideal of E .

For example, the sets $\{0\}$ and E are ideals of E . In addition, if I is an ideal of the GPEA E , then $(I; +, 0)$ is also a sub-GPEA of $(E; +, 0)$.

Theorem 3.1. *Let $(E; +, 0, 1)$ be a symmetric PEA. The following two statements are equivalent.*

- (i) *There exists a two-valued state on E .*
- (ii) *There exists a sub-GPEA $(I; +, 0)$ of $(E; +, 0)$ such that $E = \hat{I}$, I is a maximal and normal ideal of E .*

Proof. (i) \Rightarrow (ii). Assume that a mapping $s : E \rightarrow \{0, 1\}$ is a state on E . If we set $I = \text{Ker}(s)$, then $I \neq E$ is a normal ideal of E , and so, it is also a sub-GPEA of E .

We now have to prove that $E = \hat{I}$. Let $I^\#$ be the set $\{a \in E \mid s(a) = 1\}$. Since s is a two-valued state on E , we have that $I \cap I^\# = \emptyset$ and $E = I \cup I^\#$. Define the mapping $f : \hat{I} \rightarrow E$ by $f(x) = x$, $f(x^\#) = x'$ for any $x \in I$. Then f is a bijection and $f(0) = 0$, $f(0^\#) = 1$. Assume $a + b$ exists in \hat{I} for $a, b \in \hat{I}$. Then there are following three cases. (1) Both a and b belong to I , then $a + b \in I$, which implies $f(a + b) = a + b = f(a) + f(b)$. (2) Only one of a and b belongs to I , without loss of generality, assume that $a \in I$, $b \in I^\#$. Then there exists an element $c \in I$ with $b = c^\#$. By $(c \setminus a) + a + c' = 1$, we have that $f(a + b) = f(a + c^\#) = f((c \setminus a)^\#) = (c \setminus a)' = a + c' = f(a) + f(c^\#) = f(a) + f(b)$. Finally,

(3) $a, b \in \hat{I}$ but this is impossible. Hence, f is a morphism. Furthermore, assume that $f(a) \leq f(b)$ for $a, b \in \hat{I}$. There are following four cases. (1) If $a, b \in I$, then $a \leq b$ by $f(a) = a, f(b) = b$. (2) If $a, b \in I^\sharp$, then there exist $c, d \in I$ such that $a = c^\sharp, b = d^\sharp$, which imply that $c' \leq d'$, and so $d \leq c$. Hence, $a = c^\sharp \leq d^\sharp = b$. (3) If $a \in E, b \in I^\sharp$, then there exists an element $c \in I$ with $b = c^\sharp$. Therefore, $a \leq c', a \leq b$. (4) If $a \in I^\sharp, b \in I$, then there exists an element $d \in I$ with $a = d^\sharp$. Therefore, $d' \leq b, d^\sharp \leq b$, which is impossible. Hence, the statement $f(a) \leq f(b)$ implies that $a \leq b$, which implies that f is a monomorphism. Thus, f is an isomorphism between \hat{I} and E . Noticing that the set \hat{I} equals E , we have the PEA \hat{I} coincides with E .

Now, if there exists an ideal J of E , such that $I \subseteq J$ with $J \setminus I \neq \emptyset$, then there exists an element $i \in I$ such that $i' \in J$, hence, $1 \in J$, which implies that $J = E$.

(ii) \Rightarrow (i). Assume that PEA $E = I \cup I^\sharp$, then I is a symmetric GPEA by Proposition 2.9. Then define a mapping $s : E \rightarrow \{0, 1\}$ by setting $s(a) = 0, s(a^\sharp) = 1$ for any $a \in I$. Consequently, $s(0) = 0, s(1) = s(0^\sharp) = 1$. If $x, y \in E$, and $x + y$ exists in E , then $x, y \in I$ or only exactly one of x, y belongs to I . If $x, y \in I$, then $x + y \in I$, and so $s(x + y) = s(x) + s(y) = 0$. If $x \in I$ and $y \in I^\sharp$, then $x + y \in I^\sharp$, and so $s(x + y) = 1 = s(x) + s(y)$. Similarly, if $x \in I^\sharp$ and $y \in I$, then $x + y \in I^\sharp$, and so $s(x + y) = 1 = s(x) + s(y)$. Thus, s is a two-valued state on E . \square

Example 3.2. Let \mathbb{Z} be the group of integers and G be a po-group. Let $\mathbb{Z} \vec{\times} G$ be the lexicographic product of \mathbb{Z} and G . If we set $E = \Gamma(\mathbb{Z} \vec{\times} G, (1, 0))$, then E is a symmetric PEA but not necessarily commutative. Set $I = \{(0, g) \in E \mid g \in G\}$, then I is a maximal and normal ideal of E and it is routine to verify that $E = I \cup I^- = I \cup I^\sim$ and $I \cap I^- = I \cap I^\sim = \emptyset$. Thus, the symmetric PEA E admits a two-valued state, and this state is a unique state of E .

In [29, 30], Z. Riečanová and I. Marinová studied effect algebras with two-valued (discrete) states, and they proved that any effect algebra admitting a two-valued state is the unitization of a generalized sub-effect algebra. Theorem 3.1 shows that any symmetric PEA admitting a two-valued state is the unitization of a symmetric sub-GPEA, and so, it may be considered as a generalization of the results for effect algebras proved in [29]. However, for any PEA admitting a two-valued state, if it is not symmetric, then it is not a unitization of any sub-GPEA. In general, for a two-valued state PEA, we have the following result.

Theorem 3.3. *Let E be a PEA. Then E admits a two-valued state s if and only if there exists a maximal and normal ideal I such that $E = I \cup I^- = I \cup I^\sim$ and $I \cap I^- = I \cap I^\sim = \emptyset$.*

Proof. Assume that E is a PEA admitting a two-valued state s , then for any $x \in E$, either $s(x) = 0$, or $s(x) = 1$. Set $I = \text{Ker}(s)$, we have that $E = I \cup (E \setminus I)$. For any $x \in I, s(x^-) = 1$, hence, we have that $I^- \subseteq E \setminus I$. Conversely, for any $y \in E \setminus I$, we have that $s(y) = 1$, and so $s(y^\sim) = 0$, which implies $y^\sim \in I$. Noticing that $y = y^{\sim-}$, we have that $y \in I^-$, and so $E \setminus I \subseteq I^-$. Hence, $E \setminus I = I^-$. Similarly, we can prove that $E \setminus I = I^\sim$. It is obvious that $I \cap I^- = I \cap I^\sim = \emptyset$. In the same way as in Theorem 3.1, we can prove that I is a maximal and normal ideal of E .

Conversely, we assume that there exists a normal ideal I such that $E = I \cup I^- = I \cup I^\sim$ and $I \cap I^- = I \cap I^\sim = \emptyset$. Define a mapping $s : E \rightarrow \{0, 1\}$ as follows:

$$s(x) = \begin{cases} 0, & x \in I, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that s is well defined and $s(0) = 0$, $s(1) = 1$. Now, assume that $x + y$ exists in E for $x, y \in E$, then there are the following three cases:

(i) $x, y \in I$. Then $x + y \in I$, since I is an ideal of E . Therefore, $s(x + y) = s(x) + s(y) = 0$.

(ii) Only one of x and y belongs to I ; without loss of generality, we assume that $x \in I$ and $y \notin I$. Then $x + y \notin I$, since I is an ideal of E . Consequently, $s(x + y) = s(x) + s(y) = 1$.

(iii) $x \notin I$ and $y \notin I$. Now we assume that there exist $a, b \in I$ with $x = a^-$ and $y = b^-$. Then $a^- + b^-$ exists, which implies $b^- \leq a^{-\sim} = a$. Hence, $y = b^- \in I$, which is a contradiction with $y \notin I$. Thus, if $x + y$ exists in E , then at least one of x and y belongs to I .

Hence, we have proved that for $x, y \in E$, $s(x + y) = s(x) + s(y)$, whenever $x + y$ exists in E . This yields that the mapping s is a two-valued state on E . \square

Example 3.4. Let E be the PEA $\Gamma(G, (1, 0, 0))$ in Example 2.3. Assume that $s : E \rightarrow [0, 1]$ is a state on E . Notice that, for any $(0, b, c) \in E$, $n(0, b, c)$ exists in E for any $n \in \mathbb{N}$. Hence, we have $s(0, b, c) = 0$. Then $\text{Ker}(s) = \{(0, b, c) \mid (0, b, c) \in E\}$ which is a normal ideal of E . Furthermore, for any $(0, b, c) \in E$, it is easy to see that $(0, b, c)^- = (1, -b, -c)$, $(0, b, c)^\sim = (1, -c, -b)$, which implies that $E = \text{Ker}(s) \cup (\text{Ker}(s))^- = \text{Ker}(s) \cup (\text{Ker}(s))^\sim$ and $\text{Ker}(s) \cap (\text{Ker}(s))^- = \text{Ker}(s) \cap (\text{Ker}(s))^\sim = \emptyset$. Therefore, the state s of E is two-valued, and this state is a unique state of E .

4 Pseudo-effect algebras with $(n + 1)$ -valued discrete states

In this section, we give sufficient and necessary conditions in order a pseudo-effect algebra admits an $(n + 1)$ -valued state. In addition, some properties of pseudo-effect algebras having an $(n + 1)$ -valued state are studied.

Let E be a PEA and $A, B \subseteq E$. In the following, we write (i) $A \leq B$ iff $a \leq b$ for all $a \in A$, and all $b \in B$, (ii) $A + B := \{a + b \mid a \in A, b \in B \text{ and } a + b \text{ exists in } E\}$. It can happen that $a + b$ exists in E for any $a \in A$ and any $b \in B$. Then we are saying that $A + B$ exists in E .

We write $1A := A$. If $A + A$ exists, then we denote $2A = A + A$. If iA exists, and $iA + A$ exists, then we denote $(i + 1)A = iA + A$ for $i \geq 2$.

Theorem 4.1. *Let $(E; +, 0, 1)$ be a PEA. Then the following two statements are equivalent.*

- (i) *There exists an $(n + 1)$ -valued discrete state on E .*
- (ii) *There exist nonempty subsets E_0, E_1, \dots, E_n of E such that*
 - (a) *$E_i \cap E_j = \emptyset$, for any $i, j \in \{0, 1, \dots, n\}$ with $i \neq j$,*
 - (b) *$E = E_0 \cup E_1 \cup \dots \cup E_n$,*
 - (c) *$E_i^- = E_i^\sim = E_{n-i}$ for any $i \in \{0, 1, \dots, n\}$,*
 - (d) *if $x \in E_i, y \in E_j$ and $x + y$ exists in E , then $i + j \leq n$ and $x + y \in E_{i+j}$ for $i, j \in \{0, 1, \dots, n\}$.*

Proof. Assume that s is an $(n + 1)$ -valued discrete state on E , then we set $E_i = s^{-1}(\{\frac{i}{n}\})$ for any $i \in \{0, 1, \dots, n\}$. It is easy to see that (a) and (b) hold. For (c), $x \in E_i$ if and only if $s(x) = \frac{i}{n}$ if and

only if $s(x^-) = s(x^\sim) = \frac{n-i}{n}$, which entails that the statement (c) holds. For (d), assume that $x \in E_i$, $y \in E_j$ and $x+y$ exists in E , then we have that $s(x) = \frac{i}{n}$, $s(y) = \frac{j}{n}$ and $s(x+y) = s(x)+s(y) = \frac{i+j}{n} \leq 1$, which implies that $i+j \leq n$ and $x+y \in E_{i+j}$.

Conversely, define a mapping $s : E \rightarrow [0, 1]$ by $s(x) = \frac{i}{n}$ if $x \in E_i$. It is clear that s is well-defined and $s(E) = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. Take $x, y \in E$ such that $x+y$ is defined in E . Then there are unique integers i and j such that $x \in E_i$ and $y \in E_j$. By (d), we have that $i+j \leq n$ and $x+y \in E_{i+j}$. Hence, $s(x+y) = s(x) + s(y)$. Furthermore, there is a unique $i \in \{0, 1, \dots, n\}$ such that $0 \in E_i$. For any $x \in E_n$, $x+0$ and $0+x$ exist, and so, by (d) $i+n \leq n$, which implies $i=0$. Thus, $0 \in E_0$ and $1 \in E_n$ by (c). Hence, $s(0) = 0$, and $s(1) = 1$. Thus, s is an $(n+1)$ -valued discrete state on E . \square

Let E be a PEA and $n \geq 1$ be an integer. If subsets E_0, \dots, E_n of E satisfy the conditions (a)–(d) in Theorem 4.1, we say that E_0, \dots, E_n is an n -decomposition of E and we shall denote it by (E_0, \dots, E_n) . Let $\mathcal{D}_n(E) = \{(E_0, \dots, E_n) \mid (E_0, \dots, E_n) \text{ is an } n\text{-decomposition of } E\}$ and $\mathcal{S}_n(E) = \{s \mid s \text{ is an } (n+1)\text{-valued discrete state on } E\}$.

Theorem 4.2. *Let $(E; +, 0, 1)$ be a PEA and $n \geq 1$ be an integer. Then there is a bijective mapping between $\mathcal{D}_n(E)$ and $\mathcal{S}_n(E)$.*

Proof. We define a mapping $f : \mathcal{D}_n(E) \rightarrow \mathcal{S}_n(E)$ as follows: for any $D = (E_0, \dots, E_n) \in \mathcal{D}_n(E)$, $f(D) = s$, where $s : E \rightarrow [0, 1]$ is a state such that $s(E_i) = \frac{i}{n}$ for any $i \in \{0, \dots, n\}$. Assume that there exists another state s_1 on E such that $s_1(E_i) = \frac{i}{n}$ for any $i \in \{0, \dots, n\}$. For any $x \in E$, there exists a unique $i \in \{0, \dots, n\}$ such that $x \in E_i$ which implies that $s(x) = s_1(x)$. Thus, f is defined well. Now, for any $D = (E_0, \dots, E_n)$, and $D_1 = (F_0, \dots, F_n)$, if $f(D) = f(D_1) = s$, then $s(E_i) = s(F_i) = \frac{i}{n}$ for any $i \in \{0, \dots, n\}$. Hence, $s^{-1}(\{\frac{i}{n}\}) = E_i = F_i$ for any $i \in \{0, \dots, n\}$, and so $D = D_1$, which implies that f is injective. By Theorem 4.1, f is surjective. Thus, f is bijective. \square

Corollary 4.3. *Let $(E; +, 0, 1)$ be a PEA. If (E_0, E_1, \dots, E_n) is an n -decomposition of E , then E_0 is a normal ideal.*

Proof. By Theorem 4.1, there exists an $(n+1)$ -valued discrete state s such that $E_0 = \text{Ker}(s)$, and so it is a normal ideal. \square

Remark 4.4. Assume that a PEA $(E; +, 0, 1)$ admits an $(n+1)$ -valued discrete state, s , then by Theorem 4.2, there exists a unique n -decomposition (E_0, E_1, \dots, E_n) of E such that $s(E_i) = \frac{i}{n}$, $i = 0, 1, \dots, n$. We note that:

(i) For $i, j \in \{0, 1, \dots, n\}$ with $i \leq j$, $E_i \leq E_j$ does not hold in general. For example, the four element Boolean algebra $E = \{0, a, a', 1\}$ admits a 2-valued discrete state s such that $s(0) = s(a) = 0$, $s(a') = s(1) = 1$. Set $E_0 = \{0, a\}$, $E_1 = \{0, a'\}$, then $E_0 \not\leq E_1$.

(ii) In general, for $i, j \in \{0, 1, \dots, n\}$, $E_i + E_j$ does not exist when $i+j < n$. Even $E_0 + E_0$ does not exist. For example, the four element Boolean algebra $E = \{0, a, a', 1\}$ admits a two-valued state s such that $s(0) = s(a) = 0$, $s(a') = s(1) = 1$. Set $E_0 = \{0, a\}$, $E_1 = \{0, a'\}$, then $E_0 + E_0$ does not exist in E .

(iii) By Theorem 3.3, if $n = 1$, then the ideal E_0 is maximal. However, if $n \geq 2$, then E_0 is not necessarily maximal. For example, the four element Boolean algebra $E = \{0, a, a', 1\}$ admits a 3-valued discrete state s such that $s(0) = 0, s(a) = s(a') = \frac{1}{2}, s(1) = 1$. But $E_0 = \{0\}$ is not a maximal ideal of E .

Theorem 4.5. *Let (E_0, E_1, \dots, E_n) be an n -decomposition of a PEA E . Then $E_0 \leq E_1 \leq \dots \leq E_n$ if and only if $E_i + E_j$ exists in E whenever $i + j < n$ for any $i, j \in \{0, \dots, n\}$.*

In such a case,

- (i) $E_0 = \text{Infin}(E)$ and $\text{Infin}(E)$ is a normal ideal.
- (ii) $E_i + E_j = E_{i+j}$ whenever $i + j < n$.
- (iii) For any $x \in E_i, y \in E_j = E_{i+j}$, if $i + j > n$, then neither $x + y$ nor $y + x$ exists.

Proof. By Theorem 4.1, there is a unique discrete $(n + 1)$ -valued state s such that $s(E_i) = \frac{i}{n}$ for $i = 0, 1, \dots, n$.

Assume $E_0 \leq E_1 \leq \dots \leq E_n$. For any $i, j \in \{0, \dots, n - 1\}$ with $i + j < n$, we have that $i < n - j$, and so $E_i \leq E_j^-$, which implies that $E_i + E_j$ exists and so $E_i + E_j = E_{i+j}$. In fact, for any $a \in E_i, b \in E_j$, then $s(a + b) = \frac{i+j}{n}$, which implies that $a + b \in E_{i+j}$. Conversely, let $c \in E_{i+j}$. For any $a \in E_i$, we have that $a \leq c$. Then there exists an element $b \in E$ such that $a + b = c$. Whence, $s(a + b) = s(a) + s(b) = \frac{i+j}{n}$, then $s(b) = \frac{j}{n}$, which implies that $b \in E_j$. We have also proved (ii).

Conversely, let $E_i + E_j$ exist in E for $i + j < n$.

(i) For any $x, y \in E_0$, we have that $x + y$ exists in E . Then $s(x + y) = s(x) + s(y) = 0$ and $x + y \in E_0$ which implies $E_0 \subseteq \text{Infin}(E)$. Conversely, let $x \in \text{Infin}(E)$, we have that mx is defined in E for each integer $m \geq 1$. Then $s(mx) = ms(x) \leq 1$ which implies $s(x) = 0$ and $x \in \text{Ker}(s)$, and so $x \in E_0$.

For $i, j \in \{0, 1, \dots, n - 1\}$, if $i + j < n$, then $E_i + E_j$ exists in E , and so $E_i \leq E_j^- = E_{n-j}$. Now, for $i \in \{0, 1, \dots, n - 1\}$, set $j = n - i - 1$, we have that $i + j < n$, and so we have that $E_i \leq E_{n-j}^- = E_{i+1}$, which proves $E_0 \leq E_1 \leq \dots \leq E_n$.

(iii) Assume that $a \in E_i$ and $b \in E_j$ for $i + j < n$. Then $a + b$ exists and $s(a + b) = \frac{i+j}{n}$, and so $a + b \in E_{i+j}$. Conversely, let $z \in E_{i+j}$, then for any $x \in E_i, x \leq z$, so that $z = x + (x/z)$, by $s(z) = s(x) + s(x/z)$, which implies that $x/z \in E_j$.

(iv) Assume that $i + j > n, x \in E_i, y \in E_j = E_{i+j}$, either $x + y$ or $y + x$ exists, then $s(x + y) > 1$ or $s(y + x) > 1$, which is absurd. \square

Example 4.6. Let D be the set $\{0, a, b, 1\}$. Let a partial operation $+_D$ on B be defined as follows: $a +_D a = b +_D b = 1, 0 +_D a = a +_D 0 = a, 0 +_D b = b +_D 0 = b, 1 +_D 0 = 0 +_D 1 = 1$. Then the algebraic system $(D; +_D, 0, 1)$ is an effect algebra, which is usually called the diamond. Let $E_0 = \{(0, i) \mid i \in \mathbb{Z}^+\}, E_1 = \{(a, i) \mid i \in \mathbb{Z}\} \cup \{(b, j) \mid j \in \mathbb{Z}\}, E_2 = \{(1, -i) \mid i \in \mathbb{Z}^+\}$, and $E = E_0 \cup E_1 \cup E_2$. We define a partial binary operation $+$ on E as follows:

- (i) for any $x = (0, i), y = (0, j) \in E_0, x + y = (0, i + j)$.
- (ii) for any $x = (0, i) \in E_0, y = (a, j) \in E_1$, then $x + y = y + x = (a, i + j)$. For any $x = (0, i) \in E_0, z = (b, j) \in E_1$, then $x + z = z + x = (b, i + j)$.

It is routine to verify that $(E; +, 0, 1)$ is an effect algebra, where 0 and 1 denote $(0, 0)$ and $(1, 0)$, respectively. A mapping $s : E \rightarrow [0, 1]$ such that $s(E_i) = \frac{i}{2}$ for $i = 0, 1, 2$ is a 3-valued discrete state.

The following statements are true.

- (1) $E_0 = E_0 + E_0$, $E_1 = E_0 + E_1$.
- (2) Any of the following sum $E_0 + E_2$, $E_1 + E_1$, $E_1 + E_2$ does not exist.
- (3) $E_0 \leq E_1 \leq E_2$.
- (4) $E_0 = \text{Infinit}(E)$ and E_0 is a maximal ideal.

Example 4.7. Let B be the set $\{0, a, b, 1\}$. Let a partial operation $+_B$ on B be defined as follows: $a +_B b = b +_B a = 1$, $0 +_B a = a +_B 0 = a$, $0 +_B b = b +_B 0 = b$, $1 +_B 0 = 0 +_B 1 = 1$. Then the algebraic system $(B; +_B, 0, 1)$ is an effect algebra. Let (G, u) be a po-group with strong unit u . Let $E_0 = \{(0, i) \mid i \in G^+\}$, $E_1 = \{(a, i) \mid i \in G\} \cup \{(b, j) \mid j \in G\}$, $E_2 = \{(1, -i) \mid i \in G^+\}$, and $E = E_0 \cup E_1 \cup E_2$. We define a partial binary operation $+$ on E as follows:

- (i) for any $x = (0, i), y = (0, j) \in E_0$, $x + y$ exists and $x + y = (0, i + j)$,
- (ii) for any $x = (a, i), y = (b, j) \in E_1$, if $i + j \leq 0$, then $x + y$ exists and $x + y = (1, i + j)$,
- (iii) for any $x = (0, i) \in E_0, y = (a, i), z = (b, j) \in E_1$, $x + y, y + x, x + z$, and $z + x$ exist, and $x + y = (a, i + j), y + x = (a, j + i), x + z = (b, j), z + x = (b, j + i)$.

It is routine to verify that $(E; +, 0, 1)$ is a PEA, where 0 and 1 denote $(0, 0)$ and $(1, 0)$, respectively.

It is easy to see that $E_i + E_j$ exists for $i + j < 2$.

We have $E_0 = \text{Infinit}(E)$, however, E_0 is not a maximal ideal. If we set $I_a = E_0 \cup \{(a, i) \mid (a, i) \in E_1\}$ and $I_b = E_0 \cup \{(b, j) \mid (b, j) \in E_1\}$, then both I_a and I_b are proper normal ideals, and $E_0 \subsetneq I_a, E_0 \subsetneq I_b$. In fact, $\{I_a, I_b\}$ is the set of maximal ideals of E , and $E_0 = I_a \cap I_b$.

5 n -perfect PEA

We now give the definition of n -perfect PEAs as follows.

Definition 5.1. Let $(E; +, 0, 1)$ be a PEA. We say that E is an n -perfect PEA if

- (i) there exists an n -decomposition (E_0, E_1, \dots, E_n) of E .
- (ii) $E_i + E_j$ exists if $i + j < n$.
- (iii) E_0 is the unique maximal ideal of E .

We recall that according to Corollary 4.3, E_0 is a unique maximal ideal of E and it is normal.

Example 5.2. Let \mathbb{Z} be the group of integers and G be a po-group. Let $\mathbb{Z} \overrightarrow{\times} G$ be the lexicographic product of \mathbb{Z} and G , and let $u = (n, 0)$. If we set $E = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$, then E is a PEA. If we set $E_0 = \{(0, g) \mid g \in G^+\}$, for $i \in \{1, \dots, n-1\}$, $E_i = \{(i, g) \mid g \in G\}$, and $E_n = \{(0, -g) \mid g \in G^+\}$, then E is an n -perfect PEA.

We recall the following two definitions used in [14]. Let E be a PEA. We denote by $\mathcal{M}(E)$ and $\mathcal{N}(E)$ the set of maximal ideals and the set of normal ideals of E , respectively. We define (i) the *radical* of a PEA E , $\text{Rad}(E)$, as the set

$$\text{Rad}(E) = \bigcap \{I \mid I \in \mathcal{M}(E)\},$$

and (ii) the *normal radical* of E , via

$$\text{Rad}_n(E) = \bigcap \{I \mid I \in \mathcal{M}(E) \cap \mathcal{N}(E)\}.$$

It is obvious that $\text{Rad}(E) \subseteq \text{Rad}_n(E)$ holds in any PEA E .

Lemma 5.3. Let $(E; +, 0, 1)$ be an n -perfect PEA. Then $E_0 = \text{Infinit}(E) = \text{Rad}(E) = \text{Rad}_n(E)$.

Proof. By Theorem 4.5, $E_0 = \text{Infinit}(E)$. By (iii) of Definition 5.1, we have that $\text{Rad}(E) = E_0$. Furthermore, E_0 is also a normal ideal and so $\text{Rad}(E) = \text{Rad}_n(E)$. \square

Definition 5.4. An ideal I in a GPEA E is called an R_1 -ideal, if the following condition holds:

(R1) if $i \in I, a, b \in E$ and $a + b$ exists, $i \leq a + b$, then there exist $j, k \in I$ such that $j \leq a, k \leq b$ and $i \leq j + k$.

An R_1 -ideal I is called a *Riesz ideal*, if the following two conditions hold:

(R2) if $i \in I, a, b \in E, i \leq a$ and $(a \setminus i) + b$ exists, then there exists $j \in I$ such that $j \leq b$ and $a + (j/b)$ exists; if $i \in I, a, b \in E, i \leq a$ and $b + (i/a)$ exists, then there exists $j \in I$ such that $j \leq b$ and $(b \setminus j) + a$ exists.

Let A be a subset of a partially ordered set E . We say that A is *downwards* (*upwards*) directed if for any $x, y \in A$, there exists $z \in A$ such that $z \leq x, y$ ($x, y \leq z$). If E is a po-group or a PEA, then E is upwards directed iff it is downwards directed; then we say simply that E is *directed*.

Proposition 5.5. [31] *In an upwards directed GPEA E , an ideal I is a Riesz ideal if and only if I is R_1 -ideal.*

Proposition 5.6. *Let $(E; +, 0, 1)$ be an n -perfect PEA for some integer $n \geq 1$. Then E_0 is a Riesz ideal.*

Proof. Since the PEA E is upwards directed, by Proposition 5.5, it suffices to show that E_0 satisfies the condition (R1). Assume that $i \in E_0, a, b \in E, a + b$ exists, and $i \leq a + b$. There are the following three cases: (1) Both a and b belong to E_0 ; then the condition is trivial. (2) Only one of a, b belongs to E_0 , without loss of generality, we assume that $a \in E_0$ and $b \notin E_0$. By Theorem 4.5 (ii), $i \leq b$, thus $i \leq a + i \leq a + b$. (3) Neither a nor b belongs to E_0 , then by Theorem 4.5 (ii), $i \leq a, b$, and so $i \leq i + i \leq a + b$, and $i \in E_0$. \square

Definition 5.7. For an ideal I in a GPEA E , we define $a \sim_I b$ if there exist $i, j \in I, i \leq a, j \leq b$ such that $a \setminus i = b \setminus j$.

Theorem 5.8. [31] Let I be a normal Riesz ideal in a GPEA E . Then E/\sim_I is a linear GPEA if and only if I satisfies the following condition:

(L) For any $a, b \in E$, there exists a $c \in E$ such that $a + c \sim_I b$ or $b + c \sim_I a$.

Lemma 5.9. [31] If \sim is a Riesz congruence in a PEA E , then for any $a \in E$ the equivalence class $[a]$ is both upwards directed and downwards directed.

Lemma 5.10. [10] If I is an ideal of PEA E satisfying $(\text{RDP})_0$ and a is an element of E , then the ideal $I_0(I, a)$ generated by I and a is given by $I_0(I, a) = \{x \in E \mid x = x_1 + a_1 + \cdots + x_n + a_n, x_i \in I, a_i \leq a, 1 \leq i \leq n, n \geq 1\}$. If I is a normal ideal, then $I_0(I, a) = \{x \in E \mid x = x_1 + a_1 + \cdots + a_n, y \in I, a_i \leq a, 1 \leq i \leq n, n \geq 1\}$.

Proposition 5.11. Let $(E; +, 0, 1)$ be an n -perfect PEA. Then E_0 and E_n are both upwards and downwards directed.

Proof. By $0 \in E_0$, E_0 is downwards directed. For any $x, y \in E_0$, $x + y$ exists in E and $x + y \in E_0$, thus E_0 is upwards directed. By $E_n = E_0^\sim$, we have that E_n is both upwards directed and downwards directed. \square

Proposition 5.12. Let $(E; +, 0, 1)$ be an n -perfect PEA satisfying $(\text{RDP})_0$. Then E satisfies the following condition

(e) for any $i \in \{0, 1, \dots, n\}$, E_i is both upwards directed and downwards directed.

Proof. By Proposition 5.11, if $n = 1$, then the result holds. Now, assume that $n > 1$. By $E_i = E_i^- = E_i^\sim$, it suffices to prove that E_i is downwards directed, for any $i \in \{1, \dots, n-1\}$.

Assume that $x, y \in E_1$. Then the ideal $I(E_0, x)$, which is generated by the normal ideal E_0 and x , is equal to E , since E_0 is a maximal ideal. Thus, there exists $a \in E_0$, and $z_1, \dots, z_m \in E_1$ with $z_1, \dots, z_m \leq x$ such that $y = a + z_1 + \cdots + z_m$ by Lemma 5.10. By $y \in E_1$, we have that $m = 1$. Thus, $z_1 \leq x, y$ and $z_1 \in E_1$ and E_1 is downwards directed.

By Theorem 4.5, $E_i = iE_1$ for any $i \in \{1, \dots, n-1\}$. Now, assume that $x, y \in E_i$, then there exist $x_1, \dots, x_i \in E_1$, and $y_1, \dots, y_i \in E_1$, such that $x = x_1 + \cdots + x_i$ and $y = y_1 + \cdots + y_i$. Since E_1 is downwards directed, there exists a $z \in E_1$ such that $z \leq x_1, \dots, x_i$ and $z \leq y_1, \dots, y_i$. Thus, $iz \in E_i$ and $iz \leq x, y$. Hence, E_i is downwards directed for any $i \in \{1, \dots, n-1\}$. \square

For $a, b \in E$ with $a \leq b$, we define an interval $[a, b] := \{x \in E \mid a \leq x \leq b\}$.

Proposition 5.13. Let $(E; +, 0, 1)$ be an n -perfect PEA satisfying the condition (e). If any decreasing chain in E_1 has a lower bound in E_1 , then

- (i) There exists a smallest element $c \in E_1$.
- (ii) For any $i \in \{0, 1, \dots, n\}$, ic is a smallest element in E_i .
- (iii) For any $i \in \{0, 1, \dots, n\}$, $(ic)^\sim = (ic)^-$ and it is the largest element in E_{n-i} .
- (iv) For any $i \in \{0, 1, \dots, n\}$, $E_i = [ic, ((n-i)c)^\sim]$.

(v) $E = \{0, c, \dots, nc\}$.

Proof. By Zorn's Lemma, there exists a minimal element c in E_1 . Since E_1 is downwards directed, and so the minimal element c is also the smallest element in E_1 .

Now, by Theorem 4.5 (iii), for any $i \in \{1, \dots, n\}$, $E_i = iE_1$. For $i \in \{1, \dots, n\}$, $x \in E_i$, there exist $x_1, \dots, x_i \in E_1$ such that $x = x_1 + \dots + x_i$, which implies that $ic \leq x$, since c is the smallest element in E_1 . Hence, for any $i \in \{0, 1, \dots, n\}$, ic is the smallest element in E_i . Thus, $(ic)^\sim$ and $(ic)^-$ are the largest elements in E_{n-i} , and so $(ic)^\sim = (ic)^-$. Hence, $E_i = [ic, ((n-i)c)^\sim]$, for any $i \in \{0, 1, \dots, n\}$.

Since $(nc)^-$ is the largest element of E_0 , $(nc)^- + (nc)^-$ exists and $(nc)^- + (nc)^- \in E_0$, then we have that $(nc)^- = 0$, which implies that $nc = 1$. Thus, we have that $ic = ((n-i)c)^\sim$, for $i \in \{0, 1, \dots, n\}$. By $E_i = [ic, ((n-i)c)^\sim]$, we have that $E_i = \{ic\}$, $i \in \{0, 1, \dots, n\}$. \square

Recall that for any state $s : E \rightarrow [0, 1]$ on a PEA E , we can define a binary operation \sim_s as follows: $x \sim_s y$ if and only if $s(x) = s(y)$, for $x, y \in E$.

Proposition 5.14. *Let $(E; +, 0, 1)$ be an n -perfect PEA satisfying the condition (e) and let s be a state $s : E \rightarrow [0, 1]$ such that $s(E_i) = \frac{i}{n}$ for any $i \in \{0, 1, \dots, n\}$. Then:*

- (i) *For $x, y \in E$, $x \sim_s y$ if only if there exists a unique $i \in \{0, 1, \dots, n\}$ such that $x, y \in E_i$.*
- (ii) *For $x, y \in E$, $x \sim_s y$ if only if $x \sim_{E_0} y$.*

Proof. (i) It is obvious.

(ii) For $x, y \in E$, if $x \sim_{E_0} y$, then there exist $a, b \in E_0$ such that $x \setminus a = y \setminus b$, thus $s(x \setminus a) = s(y \setminus b)$. By $s(E_0) = 0$, we have that $s(x) = s(y)$, and so $x \sim_s y$.

Conversely, if $x \sim_s y$, then there is a unique $i \in \{0, 1, \dots, n\}$ such that $x, y \in E_i$. Now, if $i = 0$, then $x \sim_{E_0} y$. If $i = n$, then $1 \setminus x, 1 \setminus y \in E_0$, and so $1 \setminus x \sim_{E_0} 1 \setminus y$, hence, $x \sim_{E_0} y$. If $n = 1$, we have finished the proof. Assume that $n > 1$ and $i \in \{1, \dots, n-1\}$. Since E_i is downwards directed, there exists $z \in E_i$ such that $z \leq x, y$. Thus, $z = x \setminus (z/x) = y \setminus (z/y)$ and $z/x, z/y \in E_0$, which implies that $x \sim_{E_0} y$. \square

Proposition 5.15. *Let $(E; +, 0, 1)$ be an n -perfect PEA and a state $s : E \rightarrow [0, 1]$ such that $s(E_i) = \frac{i}{n}$ for any $i \in \{0, 1, \dots, n\}$. If for $x, y \in E$, $x \sim_s y$ if only if $x \sim_{E_0} y$, then E satisfies the condition (e).*

Proof. By the assumption, we have that for any $i \in \{0, 1, \dots, n\}$, E_i is the equivalent class of E with respect to the Riesz congruence \sim_{E_0} . Thus, by Lemma 5.9, for any $i \in \{0, 1, \dots, n\}$, E_i is both upwards directed and downwards directed. \square

Theorem 5.16. *Let $(E; +, 0, 1)$ be an n -perfect PEA satisfying the condition (e). Then E / \sim_{E_0} is isomorphic to the effect algebra $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$.*

Proof. For any $x, y \in E$, there exist unique $i, j \in \{0, 1, \dots, n\}$ such that $x \in E_i$, $y \in E_j$. Without loss of generality, we assume that $i \leq j$. Assume that $i = j$. By Theorem 4.1, there exists a state $s : E \rightarrow \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ such that $s(E_i) = \frac{i}{n}$, for $i \in \{0, 1, \dots, n\}$. By Proposition 5.14, $E / \sim_{E_0} =$

$\{E_0, E_1, \dots, E_n\}$, then $x \sim_{E_0} y$. If $i < j$, then $x < y$, and so there exists $z \in E$ such that $y = x + z$, and so $x + z \sim_{E_0} y$. By Theorem 5.8, E / \sim_{E_0} is a linear PEA.

For any $a \in E_1$, we have ma exists and $ma \in E_m$ for $m \in \{1, \dots, n-1\}$ by Definition 5.1. Now, $(n-1)a + ((n-1)a)^\sim = 1$, and $((n-1)a)^\sim \in E_1$. However, since E_1 is downwards directed, and so, there exists an element $c \in E_1$ such that $c \leq a, ((n-1)a)^\sim$. Whence, for $i \in \{0, 1, \dots, n\}$, ic exists and $ic \in E_i$. Thus, $E_i = (ic) / \sim_{E_0}$, for $i \in \{0, 1, \dots, n\}$. Hence, we can define the mapping $\phi : E / \sim_{E_0} \rightarrow \{0, \frac{1}{n}, \dots, 1\}$ by $\phi(E_i) = \frac{i}{n}$ for any $i \in \{0, 1, \dots, n\}$, which is an isomorphism between effect algebras. \square

6 Representation of strong n -perfect PEA

In [33], the author studied the structure of non-Archimedean effect algebras and gave some conditions such that a non-Archimedean effect algebra E is isomorphic to the lexicographical product of one Archimedean effect algebra with a linearly ordered group. We recall that a PEA E is *Archimedean* if $\text{Infin}(E) = \{0\}$.

In this section, we introduce a stronger class of n -perfect PEAs, called strong n -perfect PEAs. We will give conditions such that any strong n -perfect PEA is isomorphic with the n -perfect PEA $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$, where G is a torsion-free po-group such that $\mathbb{Z} \overrightarrow{\times} G$ satisfies $(\text{RDP})_1$. In addition, we will study a categorical equivalence of the category of strong n -perfect PEAs with a special category of torsion-free directed po-groups.

Let $(E; +, 0, 1)$ be a PEA and $(G; +, \leq)$ be a directed po-group with a fixed element $h \in G$. Let $E \overrightarrow{\times}_h G$ be the set $\{(0, g) \mid g \in G^+\} \cup \{(a, g) \mid a \in E \setminus \{0, 1\}, g \in G\} \cup \{(1, g) \mid g \leq h, g \in G\}$, and define a partial addition $+$ on $E \overrightarrow{\times}_h G$ componentwise just as following, for any $(a, x), (b, y), (c, z) \in E \overrightarrow{\times}_h G$, $(a, x) + (b, y)$ exists and equals to (c, z) if and only if $a + b = c, x + y = z$. It is routine to prove that $(E \overrightarrow{\times}_h G; +, (0, 0), (1, h))$ is a pseudo-effect algebra. The set $(E \overrightarrow{\times}_h G; +, (0, 0), (1, h))$ is called the *lexicographical product* of the pseudo-effect algebra E with the po-group G and with respect to the element h of G . We recall that $E \overrightarrow{\times}_h G$ can be also expressed via the Γ functor as $\Gamma(E \overrightarrow{\times} G, (1, h)) := \{(a, g) \mid (0, 0) \leq (a, g) \leq (1, h)\}$.

It is routine to verify the following proposition.

Proposition 6.1. *Let $(E; +, 0, 1)$ be the effect algebra $\{0, \frac{1}{n}, \dots, 1\}$ and $(G; +, \leq)$ be a directed po-group. Then the lexicographical product $(E \overrightarrow{\times}_h G; +, (0, 0), (1, h))$ of the effect algebra E and the po-group G with respect to the element h is the n -perfect PEA $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, h))$.*

Proposition 6.2. *Let $(E; +, 0, 1)$ be an n -perfect PEA. Then there exists a unique directed po-group G such that $G^+ = E_0$.*

Proof. By Theorem 4.5, $E_0 = \text{Infin}(E)$. Furthermore, $E_0 + E_0$ exists and $E_0 + E_0 = E_0$. Hence, for any $x, y \in E_0$, $x + y \in E_0$, and $(E_0; +, 0)$ is a semigroup. For any $x, y \in E_0$, the equation $x + y = 0$, implies that $x = y = 0$. For any $x, y, z \in E_0$, the equation $x + y = x + z$ implies that $y = z$, and equation $y + x = z + x$ implies that $y = z$. Then $(E_0; +, 0)$ is a cancellative semigroup satisfying the

conditions of Birkhoff, [21, Thm II.4], which guarantees that E_0 is the positive cone of a unique (up to isomorphism) po-group G . Without loss of generality, we can assume that G is generated by the positive cone E_0 , so that G is directed, see [21, Prop II.5]. \square

Proposition 6.3. *Let $(E; +, 0, 1)$ be an n -perfect PEA and H be a partially ordered group with strong unit u . Assume that $E = \Gamma(H, u)$. Then the following statement holds.*

(*) *For $x, y \in E \setminus E_0$, $a, b, c, d, e, f, g, h \in E_0$, if $x \setminus a = y \setminus b$ and $x \setminus c = y \setminus d$, then $b/a = d/c$ and $a/b = c/d$ hold in G . If $e/x = f/y$ and $g/x = h/y$, then $e \setminus f = g \setminus h$ and $f \setminus e = h \setminus g$ hold in H .*

Proof. We assume that (E_0, \dots, E_n) is an n -decomposition of E and G is a unique po-group determined by E such that $G^+ = E_0$. Since E is an interval PEA, we assume there exists a positive element u of a po-group $(H; +, 0)$ such that $\Gamma(H, u) = E$. Thus, G is a subgroup of H with $G^+ \subseteq H^+$.

If $x \setminus a = y \setminus b$ and $x \setminus c = y \setminus d$, then we have that $x = y + (-b) + a = y + (-d) + c$, $y = x \setminus a + b = x \setminus c + d$ and so $-y + x = (-b) + a = (-d) + c$, $-x + y = -a + b = -c + d$, which implies $y/x = b/a = d/c$, $x/y = a/b = c/d$. The proof of the rest is similar. \square

Proposition 6.4. *Let $(E; +, 0, 1)$ be an n -perfect PEA with an n -decomposition (E_0, \dots, E_n) . If there exists an element $c \in E_1$ such that $nc = 1$, then, for any $x \in E$, $x + c$ exists if and only if $c + x$ exists.*

Proof. Since $nc = 1$, we have that $c^\sim = c^-$. Then $x + c$ exists if and only if $x \leq c^-$ if and only if $x \leq c^\sim$ if and only if $c + x$ exists. \square

The following notions were defined for GMV-algebras in [14], and cyclic elements were defined also in [15].

Let $n > 0$ be an integer. An element a of a PEA E is said to be *cyclic of order $n > 0$* if na exists in E and $na = 1$. If a is a cyclic element of order n , then $a^- = a^\sim$, indeed, $a^- = (n-1)a = a^\sim$.

We say that a group G is *torsion-free* if $ng \neq 0$ for any $g \neq 0$ and every nonzero integer n . For example, every ℓ -group is torsion-free, see [23, Cor 2.1.3]. We recall that if G is torsion-free, so is $\mathbb{Z} \overrightarrow{\times} G$.

We recall that a group G enjoys *unique extraction of roots* if, for all positive integers n and $g, h \in G$, $g^n = h^n$ implies $g = h$. We recall that every linearly ordered group, or a representable ℓ -group, in particular every Abelian ℓ -group enjoys unique extraction of roots, see [23, Lem. 2.1.4].

We say that a PEA E enjoys *unique extraction of roots of 1* if $a, b \in E$ and na, nb exist in E , and $na = 1 = nb$, then $a = b$. Then every $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ enjoys unique extraction of roots of 1 for any $n \geq 1$ and any torsion-free directed po-group G . Indeed, let $k(i, g) = (n, 0) = k(j, h)$. Then $ki = n = kj$ which yields $i = j > 0$, and $kg = 0 = kh$ implies $g = 0 = h$.

Definition 6.5. Let E be an n -perfect PEA satisfying $(\text{RDP})_1$. We say that E is a *strong n -perfect PEA* if

- (i) there exists a torsion-free unital po-group (H, u) such that $E = \Gamma(H, u)$,
- (ii) there exists an element $c \in E_1$ such that (a) $nc = u$, and (b) $c \in C(H)$.

The element c from (ii) is said to be a *strong cyclic element of order n* .

Lemma 6.6. *Any strong cyclic element c of order n is a unique element $d \in E = \Gamma(H, u)$ such that $nd = u$ implies $c = d$ whenever H is torsion-free.*

Proof. Indeed, since $c \in C(H)$ and $d \in H$, we have $c + d = d + c$ in the group H . Then $n(c - d) = nc - nd = 0$ so that $c = d$. \square

Theorem 6.7. *Let E be a PEA and let $n \geq 1$ be an integer. Then E is a strong n -perfect PEA if and only if there exists a torsion-free directed po-group G such that $\mathbb{Z} \overrightarrow{\times} G$ satisfies $(\text{RDP})_1$, and E is isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$.*

If it is a case, G is unique and satisfies $(\text{RDP})_1$.

Proof. If there exists a torsion-free directed po-group G such that E is isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$, then E is an n -perfect PEA with a unique strong cyclic element $(1, 0)$ of order n . Hence, E is a strong n -perfect PEA.

Conversely, assume that E is a strong n -perfect PEA with an n -decomposition (E_0, \dots, E_n) and $E = \Gamma(H, u)$ for a torsion-free unital po-group (H, u) satisfying $(\text{RDP})_1$. By [18, Thm 5.7], (H, u) is a unique (up to isomorphism of unital po-groups) unital po-group with $(\text{RDP})_1$. By Lemma 6.6, there exists a unique strong cyclic element $c \in E_1$ such that $nc = u$ and $c + g = g + c$ for any $g \in H$. Thus, $E_i = (ic)/\sim_{E_0}$ for $i \in \{0, 1, \dots, n\}$. Furthermore, by Proposition 6.2, there exists a directed po-group G such that G is a subgroup of H and $E_0 = G^+ \subseteq H^+$.

We define a mapping $\varphi : E \rightarrow \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ as follows, if $x \in E_i$, then $\varphi(x) = (i, (ic)/x)$, $i \in \{0, 1, \dots, n\}$. Since E is an interval PEA, by Proposition 6.3, the condition $(*)$ holds, which implies that ϕ is defined well, and $\varphi(0) = (0, 0)$, $\varphi(u) = (n, 0)$.

Assume $x + y$ exists in E for $x, y \in E$. Then there exist unique $i, j \in \{0, 1, \dots, n\}$ such that $x \in E_i$, $y \in E_j$, and so $x + y \in E_{i+j}$. By the definition of φ , we have that $\varphi(x + y) = (i + j, ((i + j)c)/(x + y))$. Since for any $g \in H$, $c + g = g + c$, we have that $-c + g = g - c$, which implies $((i + j)c)/(x + y) = (ic + jc)/(x + y) = -(ic + jc) + x + y = -jc - ic + x + y = -ic + x - jc + y = (ic)/x + (jc)/y$, and so $(i + j, ((i + j)c)/(x + y)) = (i + j, (ic)/x + (jc)/y) = (i, (ic)/x) + (j, (jc)/y)$, which implies $\varphi(x + y) = \varphi(x) + \varphi(y)$. Thus, φ is a morphism between pseudo-effect algebras. Assume $\varphi(x) = (i, (ic)/x)$, $\varphi(y) = (j, (jc)/y)$, and $\varphi(x) \leq \varphi(y)$. There are the following two cases. (i) If $i = j$, then $(ic)/x \leq (jc)/y$, and so $x \leq y$. (ii) If $i < j$, then $x \in E_i$, $y \in E_j$, which implies that $x < y$ by Theorem 4.5 (ii). Thus, φ is a monomorphism. For any $g \in G^+$, $\varphi^{-1}((0, g)) = g \in E_0$, $\varphi^{-1}((n, -g)) = g^- \in E_n$. Assume that $(i, g) \in \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ and $i \in \{1, \dots, n - 1\}$. Then $ic + g = \varphi^{-1}((i, g))$. Thus, φ is surjective.

Hence, φ is an isomorphism from the strong n -perfect pseudo-effect algebra E onto the PEA $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$. Since (H, u) is a unique unital po-group with $(\text{RDP})_1$ such that $E = \Gamma(H, u)$ and E is isomorphic with $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$, we have by [18, Thm 5.7] that (H, u) and $(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ are isomorphic unital po-groups with $(\text{RDP})_1$ and whence, $\mathbb{Z} \overrightarrow{\times} G$ is torsion-free. We show that also G is torsion-free. Indeed, assume that, for some integer $n \neq 0$ and some element $g \in G$, we have $ng = 0$. But ng belongs also to the torsion-free po-group H_g whence, $g = 0$.

Finally, since $\mathbb{Z} \overrightarrow{\times} G$ satisfies $(\text{RDP})_1$, then clearly so does G . \square

It is worthy to recall that we do not know whether if a directed po-group G has $(\text{RDP})_1$, does have $(\text{RDP})_1$ also $\mathbb{Z} \overrightarrow{\times} G$? This is know only for Abelian po-group, see [24, Cor 2.12].

Corollary 6.8. *Let E be a strong n -perfect PEA. Then:*

- (i) *There exists a unique strong cyclic element of order n in E .*
- (ii) *There exists a unique n -decomposition (E_0, \dots, E_n) of E .*
- (iii) *The state $s : E \rightarrow [0, 1]$ such that $s(E_i) = \frac{i}{n}$ for $i \in \{0, \dots, n\}$ is extremal.*
- (iv) *E is a symmetric PEA.*

Proof. By Theorem 6.7, there exists a torsion-free directed po-group G such that E is isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$.

(i) The element $(1, 0)$ is a unique strong cyclic element of order n of $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$, which implies that there exists a unique strong cyclic element of order n in E , see Lemma 6.6.

(ii) The pseudo-effect algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ admits a unique n -decomposition (E_0, \dots, E_n) , where $E_0 = \{(0, g) \mid g \in G^+\}$, $E_n = \{(n, -g) \mid g \in G^+\}$, $E_i = \{(i, g) \mid g \in G\}$, for $i \in \{0, \dots, n-1\}$.

(iii) It easy to see that a function $s : \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0)) \rightarrow [0, 1]$ such that $s(i, g) = \frac{i}{n}$ for $i \in \{0, \dots, n\}$ is a unique state on $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$. Indeed, let s_1 be a state on $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$. It is clear that $E_0 = \text{Infinit}(E) \subseteq \text{Ker}(s_1)$. On the other hand $\text{Ker}(s_1) \subseteq E_0$ because, E_0 is a maximal ideal, which yields $\text{Ker}(s_1) = E_0$. Moreover, $1 = s_1(n(1, 0)) = ns_1(1, 0)$ which gives $s_1(1, 0) = 1/n$. Let $g \geq 0$, then $(1, g) = (1, 0) + (0, g)$ which yields $s_1(1, g) = 1/n$. Let $g \in G$ be arbitrary. Since G is directed, every element $g = g_1 - g_2$ for some $g_1, g_2 \geq 0$. Hence, $(1, g) \leq (1, g_1)$ which entails $s_1(1, g) \leq \frac{1}{n}$, and similarly, $s_1(i, g) \leq \frac{i}{n}$ for $i = 1, \dots, n-1$. Therefore, $1 = s_1(1, g) + s_1(n-1, -g) \leq \frac{1}{n} + \frac{n-1}{n} = 1$ which implies $s_1(1, g) = \frac{1}{n}$ and $s_1(E_i) = \frac{i}{n}$ for any $i = 0, 1, \dots, n$.

Hence, E admits a unique state s such that $s(E_i) = \frac{i}{n}$ for $i \in \{0, \dots, n\}$, and so it is also extremal.

(iv) The PEA $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ is a symmetric PEA, and so E is also symmetric. \square

Theorem 6.9. *Let E and F be two strong n -perfect PEAs, and (E_0, \dots, E_n) and (F_0, \dots, F_n) be n -decompositions of E and F , respectively. If $f : E \rightarrow F$ is a homomorphism between E and F and if G and H are two directed po-groups which are determined by E and F , respectively, by the property $G^+ = E_0$ and $H^+ = F_0$, then*

- (i) *$f(E_i) \subseteq F_i$, for $i \in \{0, 1, \dots, n\}$,*
- (ii) *there exists a unique homomorphism $\widehat{f} : G \rightarrow H$ such that for any $g \in G^+$, $\widehat{f}(g) = h$ iff $f(g) = h$.*

Proof. Since E and F are two strong n -perfect PEAs, by Theorem 6.7, there exist unique directed torsion-free po-groups G and H with $(\text{RDP})_1$ such that E and F are isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ and $\Gamma(\mathbb{Z} \overrightarrow{\times} H, (n, 0))$. Thus, we can assume that $E = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ and $F = \Gamma(\mathbb{Z} \overrightarrow{\times} H, (n, 0))$, and $E_0 = \{(0, g) \mid g \in G^+\}$, $E_n = \{(0, -g) \mid g \in G^+\}$, $F_0 = \{(0, h) \mid h \in H^+\}$, $F_n = \{(0, -h) \mid h \in H^+\}$, and for any $i \in \{1, \dots, n-1\}$, $E_i = \{(i, g) \mid g \in G\}$, $F_i = \{(i, h) \mid h \in H\}$.

(i) For any $(0, g) \in E_0 = \text{Infin}(E)$ and any integer $k \geq 1$, $k(0, g) = (0, kg) \in E_0$, we have $f(0, kg) = kf(0, g)$, we conclude $f(0, g) \in F_0 = \text{Infin}(F)$. Thus, we have that $f(E_0) \subseteq F_0$. Further, by $E_n = E_0^-$ and $F_n = F_0^-$, we have that $f(E_n) \subseteq F_n$.

Assume $n > 1$. For $(1, 0) \in E_1$, by $n(1, 0) = (n, 0)$, we have that $nf(1, 0) = (n, 0)$, which implies that $f(1, 0) \in F_1$. For any $(1, g)$, there exists $g_1, g_2 \in G^+$ such that $g = g_1 - g_2$ and so $(1, g) = (0, g_1) + (1, -g_2)$. Now, $f(1, -g_2) \leq f(1, 0)$ which entails $f(1, -g_2) \in F_0 \cup F_1$. But $f(1, 0) = f(1, -g_2) + f(0, g_2) \in F_1$ and $f(0, g_2) \in F_0$, so that $f(1, -g_2) \in F_1$. Consequently, $f(1, g) = f(0, g_1) + f(1, -g_2) \in F_1$ for any $g \in G$.

In the same way, we can show that $f(i, g) \in F_j$ for some $j = 0, 1, \dots, i < n$. In any rate, we state $f(1, g) \in F_1$. If not, then $f(1, g) \in F_0$ and $f(n-1, g) \in F_j$ for some $j = 0, 1, \dots, i$. But $f(n, 0) \in F_n$ and $f(n, 0) = f(1, g) + f(n-1, -g) \in F_0 + F_j = F_j \subseteq E \setminus F_n$, which is absurd.

Since for any $i \in \{1, \dots, n-1\}$, $E_i = iE_1$, and so we have that $f(E_i) \subseteq F_i$.

(ii) We define $f_1 : G^+ \rightarrow H^+$ as follows: for $g \in G^+$, $f_1(g) = h$ iff $f(0, g) = (0, h)$. Obviously, f_1 is defined well and $f(0, g) = (0, f_1(g))$ for $g \in G^+$. Furthermore, for any $g_1, g_2 \in G^+$, by $f(0, g_1 + g_2) = f(0, g_1) + f(0, g_2)$ and $f(0, g) = (0, f_1(g))$, we have that $f_1(g_1 + g_2) = f_1(g_1) + f_1(g_2)$.

We now define $\hat{f} : G \rightarrow H$ as follows: for any $g \in G$, we $\hat{f}(g) = f_1(g_1) - f_1(g_2)$ whenever $g = g_1 - g_2$, where $g_1, g_2 \geq 0$. We assert that \hat{f} is a well-defined mapping. Indeed, if $g = -h_1 + h_2$ for some $h_1, h_2 \geq 0$, then $g = g_1 - g_2 = -h_1 + h_2$, and $h_1 + g_1 = h_2 + g_2$, which implies that $f_1(h_1 + g_1) = f_1(h_2 + g_2)$, and so $f_1(g_1) - f_1(g_2) = -f_1(h_1) + f_1(h_2)$. Thus, \hat{f} is defined well.

For $g, h \in G$, we want to verify that $\hat{f}(g + h) = \hat{f}(g) + \hat{f}(h)$. We assume that $g = -g_1 + g_2$, $h = h_1 - h_2$, and $g + h = k_1 - k_2$, then $g_2 + h_1 - h_2 = g_1 + k_1 - k_2$, which entails that $g_2 + h_1 - h_2 = g_1 + k_1 - k_2$, and so, $f_1(g_2) + f_1(h_1) - f_1(h_2) = f_1(g_1) + f_1(k_1) - f_1(k_2)$, hence, $\hat{f}(g + h) = \hat{f}(g) + \hat{f}(h)$. Thus, \hat{f} is a group homomorphism. Moreover, if $g \geq 0$, then $\hat{f}(g) \geq 0$. Furthermore, by the definition of \hat{f} , we have that $\hat{f}(G^+) \subseteq H^+$, and $g \in G^+$, $\hat{f}(g) = h$.

Now, if $k : G \rightarrow H$ is a homomorphism such that for $g \in G^+$, $\hat{f}(g) = h$ iff $f(0, g) = (0, h)$. Then $\hat{f}|_{G^+} = k|_{G^+}$, since G is directed, we have that $\hat{f} = k$. \square

Let \mathcal{G} be the category whose objects are torsion-free directed po-groups G such that $\mathbb{Z} \overrightarrow{\times} G$ satisfies $(\text{RDP})_1$ and morphisms are po-group homomorphisms. Let \mathcal{SPPEA}_n be the category whose objects are strong n -perfect PEAs and morphisms are homomorphisms of PEAs.

We define a functor $\mathcal{E}_n : \mathcal{G} \rightarrow \mathcal{SPPEA}_n$ as follows: for $G \in \mathcal{G}$, let $\mathcal{E}_n(G) := \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$ and if h is a group homomorphism with domain G , we set

$$\mathcal{E}_n(h)(x) = (i, h((ic)/x)),$$

where c is a unique strong cyclic element of order n in E .

Theorem 6.10. *The functor \mathcal{E}_n is a faithful and full functor from the category \mathcal{G} of directed po-groups into the category \mathcal{SPPEA}_n of n -perfect PEAs.*

Proof. Let h_1 and h_2 be two morphisms from G_1 into G_2 such that $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. Since both G_1 and G_2 are directed, it suffice to prove that $h_1(g) = h_2(g)$ for all $g \in G_1^+$. By $\mathcal{E}(h_1) = \mathcal{E}(h_2)$, then $(0, h_1(g)) = (0, h_2(g))$ for all $g \in G_1^+$, and hence $h_1 = h_2$. \square

Let $f : \Gamma(\mathbb{Z} \overrightarrow{\times} G_1, (n, 0)) \rightarrow \Gamma(\mathbb{Z} \overrightarrow{\times} G_2, (n, 0))$ be a PEA homomorphism. Then for any $x \in G_1^+$, there exists a unique $y \in G_2^+$ such that $f(0, x) = (0, y)$. Define a mapping $h : G_1^+ \rightarrow G_2^+$ by $h(x) = y$ iff $f(0, x) = (0, y)$. Note for any $x_1, x_2 \in G_1^+$, $h(x_1 + x_2) = h(x_1) + h(x_2)$. Since G_1 is directed, for any $x \in G_1$, there exists $x_1, x_2, g_1, g_2 \in G_1^+$ such that $x = x_1 - x_2$ and $x = -g_1 + g_2$, then $g_1 + x_1 = g_2 + x_2$, and so $h(g_1) + h(x_1) = h(g_2) + h(x_2)$, which implies that $h(x_1) - h(x_2) = -h(g_1) + h(g_2)$. This shows that the assignment $h(x) = h(x_1) - h(x_2)$ is a well-defined extension of h to the whole directed po-group G_1 , and h is a po-group homomorphism. \square

We say that a *universal* group for a PEA E is a pair (G, γ) consisting of a directed po-group G and a G -valued measure $\gamma : E \rightarrow G$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ exists in E) such that the following conditions hold: (i) $\gamma(E)$ generates G , and (ii) if H is a group and $\phi : E \rightarrow H$ is an H -valued measure, then there exists a (unique) group homomorphism $\phi^* : G \rightarrow H$ such that $\phi = \phi^* \circ \gamma$.

Theorem 6.11. *Let E be a strong n -perfect PEA. Then the directed po-group $\mathbb{Z} \overrightarrow{\times} G$ from Theorem 6.7 together with the isomorphism $\gamma : E \rightarrow \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0)) \subset \mathbb{Z} \overrightarrow{\times} G$ is a universal group of E .*

Proof. Let E be a strong n -perfect PEA. By Theorem 6.7, there is a unique torsion-free directed po-group G such that E is isomorphic with $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$. Set $\mathcal{G} = \mathbb{Z} \overrightarrow{\times} G$, and $\gamma : E \rightarrow \mathbb{Z} \overrightarrow{\times} G$ be the embedding mapping, then:

(i) $\gamma(E)$ generates the group \mathcal{G} and because $(1, 0)$ is a strong unit, \mathcal{G} is directed.

(ii) Assume $\phi : E \rightarrow K$ is a K -valued measure. Then $\phi(0, 0) = 0_H$. Notice that E is symmetric, and so for any $g \in G^+$, $\phi(1, -g) = \phi((1, 0) \setminus (0, g)) = \phi((0, g) / (1, 0))$, which implies that $\phi(1, 0) - \phi((0, g) = -\phi((0, g) + \phi(1, 0)$. Define a mapping $\phi^* : \mathcal{G} \rightarrow H$, as follows, for any $g, h \in G^+$,

- (a) $\phi^*(0, g) = \phi(0, g)$,
- (b) $\phi^*(0, -g) = -\phi(0, g)$,
- (c) $\phi^*(0, g - h) = \phi(0, g) - \phi(0, h)$,
- (d) $\phi^*(0, -g + h) = -\phi(0, g) + \phi(0, h)$,
- (e) $\phi^*(1, g - h) = \phi(1, 0) + \phi^*(0, g - h)$,
- (f) $\phi^*(m, g - h) = m\phi(1, 0) + \phi^*(0, g - h)$.

For $g \in G$, if there exist $g_1, g_2, h_1, h_2 \in G^+$, such that $(0, g) = (0, g_1 - g_2) = (0, h_1 - h_2)$, then there exist $k_1, k_2 \in G^+$, such that $g = -k_1 + k_2$, since G is a directed po-group. Thus, we have that $k_1 + g_1 = k_2 + g_2$, $k_1 + h_1 = k_2 + h_2$, which implies that $\phi(0, k_1) + \phi(0, g_1) = \phi(0, k_2) + \phi(0, g_2)$, $\phi(0, k_1) + \phi(0, h_1) = \phi(0, k_2) + \phi(0, h_2)$, thus, $-\phi(0, k_1) + \phi(0, k_2) = \phi(0, g_1) - \phi(0, g_2) = \phi(0, h_1) - \phi(0, h_2)$. Consequently, ϕ^* is defined well.

For any $g_1, g_2, h_1, h_2 \in G^+$, there exists $k_1, k_2 \in G^+$ such that $-g_2 + h_1 = k_1 - k_2$, and so $(0, g_1 - g_2) + (0, h_1 - h_2) = (0, g_1 + k_1 - k_2 - h_2)$. Hence, $\phi^*((0, g_1 - g_2) + (0, h_1 - h_2)) = \phi^*(0, g_1 + k_1 - k_2 - h_2) = \phi(0, g_1 + k_1) - \phi(0, h_2 + k_2) = \phi(0, g_1) + \phi(0, k_1) - \phi(0, k_2) - \phi(0, h_2) = \phi(0, g_1 + k_1) - \phi(0, h_2 + k_2) = \phi(0, g_1) + \phi(0, k_1) - \phi(0, k_2) - \phi(0, h_2) = \phi(0, g_1) + \phi^*(0, k_1 - k_2) - \phi(0, h_2) = \phi(0, g_1) + \phi^*(0, -g_2 + h_1) - \phi(0, h_2) = \phi(0, g_1) - \phi(0, g_2) + \phi(0, h_1) - \phi(0, h_2) = \phi(0, g_1) - \phi(0, g_2) + \phi(0, h_1) - \phi(0, h_2) = \phi^*(0, g_1 - g_2) + \phi^*(0, h_1 - h_2).$

Since for any $g \in G^+$, we have that $\phi(1, 0) - \phi(0, g) = -\phi(0, g) + \phi(1, 0)$, which implies that $\phi(0, g) + \phi(1, 0) - \phi(0, g) = \phi(1, 0)$, and so $\phi(0, g) + \phi(1, 0) = \phi(1, 0) + \phi(0, g)$. Thus, for any $g \in G$, we have that $\phi(1, 0) + \phi^*(0, g) = \phi^*(0, g) + \phi(1, 0)$.

For $g_1, g_2, h_1, h_2 \in G^+$, $(i, g_1 - g_2) + (j, h_1 - h_2) = (i + j, g_1 - g_2 + h_1 - h_2)$, and so, $\phi^*((i, g_1 - g_2) + (j, h_1 - h_2)) = \phi^*(i + j, g_1 - g_2 + h_1 - h_2) = (i + j)\phi(1, 0) + \phi^*(0, g_1 - g_2 + h_1 - h_2) = i\phi(1, 0) + j\phi(1, 0) + \phi^*(0, g_1 - g_2) + \phi^*(0, h_1 - h_2) = i\phi(1, 0) + \phi^*(0, g_1 - g_2) + j\phi(1, 0) + \phi^*(0, h_1 - h_2) = \phi^*(i, g_1 - g_2) + \phi^*(j, h_1 - h_2)$.

Thus, ϕ^* is a group homomorphism with $\phi = \phi^* \circ \gamma$. \square

Recall that a functor \mathcal{E} from a category \mathcal{A} into a category \mathcal{B} is said to be *left-adjoint* provided that for every \mathcal{B} -object B there exists an \mathcal{E} -universal arrow with domain B , see [1].

Theorem 6.12. *The functor \mathcal{E}_n has a left-adjoint.*

Proof. By Theorem 6.7, for any strong n -perfect PEA E , there exists a unique torsion-free directed po-group G such that $\mathbb{Z} \overrightarrow{\times} G$ has $(\text{RDP})_1$, and by Theorem 6.11, $(\mathbb{Z} \overrightarrow{\times} G, \gamma)$ is a universal group for E .

Now, for a strong n -perfect PEA $F = \mathcal{E}_n(G_1)$, where G_1 is a torsion-free directed po-group such that $\mathbb{Z} \overrightarrow{\times} G$ has $(\text{RDP})_1$, assume that $f' : E \rightarrow \mathcal{E}_n(G_1)$ is a homomorphism between PEAs. There exists a unique group homomorphism $f_1 : \mathbb{Z} \overrightarrow{\times} G \rightarrow \mathbb{Z} \overrightarrow{\times} G_1$ such that $f' = f_1 \circ \gamma$. Now, we define $f : G \rightarrow G_1$ as $f(g) = h$ iff $f_1(0, g) = (0, h)$ for any $g \in G^+$, since G is directed, it is routine to verify that f is a homomorphism between G and G_1 . By Theorem 6.9, f is a unique homomorphism between G and G_1 such that $f(g) = h$ iff $f_1(0, g) = (0, h)$ for any $g \in G^+$, which implies that it is also a unique homomorphism such that $f' = \mathcal{E}(f) \circ \gamma$. \square

We define a functor $\mathcal{P}_n : \mathcal{SPPEA}_n \rightarrow \mathcal{G}$ as follows: if $G \in \mathcal{G}$, let

$$\mathcal{P}_n(\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))) := G,$$

where $(\mathbb{Z} \overrightarrow{\times} G, \gamma)$ is a universal group of the strong n -perfect PEA $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (n, 0))$.

Theorem 6.13. *The functor \mathcal{P}_n is a left-adjoint of the functor \mathcal{E}_n .*

Proof. It follows from Theorem 6.12 and the definition of \mathcal{P}_n . \square

Recall that a functor \mathcal{F} from a category \mathcal{A} into a category \mathcal{B} is called a *categorical equivalence* provided that it is full, faithful, and isomorphism-dense in the sense that for any \mathcal{B} -object B there exists some \mathcal{A} -object A such that $\mathcal{F}(A)$ is isomorphic to B , see [1].

Theorem 6.14. *The functor \mathcal{E}_n is a categorical equivalence of the category \mathcal{G} of directed torsion-free po-groups G such that $\mathbb{Z} \overrightarrow{\times} G$ has $(\text{RDP})_1$ and the category \mathcal{SPPEA}_n of n -strong perfect pseudo-effect algebras.*

Proof. It suffices to prove that for a strong n -perfect pseudo-effect algebra E , there is a torsion-free directed po-group G such that $\mathbb{Z} \overrightarrow{\times} G$ has $(\text{RDP})_1$ and such that $\mathcal{E}_n(G)$ is isomorphic to E . To show that, we take a universal group $(\mathbb{Z} \overrightarrow{\times} G, \gamma)$. Then $\mathcal{E}_n(G)$ and E are isomorphic. \square

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